

THE PYRAMID DISTRIBUTION

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ABSTRACT. The paper introduces the pyramid probability distribution through its density in two dimensions, and investigates its properties and those of its copula. The research focuses on ways in which the pyramid distribution demonstrates dependence between its variables, primarily as revealed by its copula and related functions. The pyramid distribution bears an intimate relationship to the normal distribution, a relationship revealed and investigated. The pyramid density is built from the normal distribution function, making the pyramid the normal distribution once removed. Having normal margins, the pyramid returns to its foundation. The paper presents a general theory of this distribution, some formal, some discursive, including the presentation of a one-parameter family.

1. INTRODUCTION

The paper introduces the pyramid probability distribution through its density in two dimensions, and investigates its properties and those of its copula. The distribution has normal margins, though it is not binormal. Properties the two distributions share are zero values in several statistics of concordance: Pearson's product-moment correlation, Kendall's tau, Spearman's rho, and Blomqvist's beta. As well, both distributions exhibit tail independence. The research, therefore, focuses on ways in which the pyramid distribution demonstrates dependence between its variables, primarily as revealed by its copula and related functions.

The attractiveness of the pyramid distribution lies in its intimate relationship to the normal distribution. In brief, the pyramid density is built from the normal distribution function, making the pyramid, in a sense, the normal distribution once removed. The pyramid, having normal margins, in effect returns to its foundation. Many interesting questions arise about what other relationships between probability distributions, such as this, may appear upon examination, stimulating thought about a general theory. Some of these ideas come to the fore in Section 4 on a general transformation and again in Section 9 on conclusions. The task at hand, however, is to investigate this specific distribution in a quest to understand its nature.

Following a section on motivation, which discusses a discrete version of the pyramid distribution, the development progresses through a description of the density, to the correlation of the variables, and to the distribution function with a determination of Kendall's tau. The

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discussion then turns to the general transformation operating on an arbitrary density producing another, the latter being marginal to a bivariate distribution constructed in the manner of the pyramid. After that comes a calculation of the characteristic function, and brief treatments on infinite divisibility and scalability. The functional form of the copula, including its quality of symmetric dependence, as defined, succeed. Calculation of Spearman's rho and Blomqvist's beta come next, along with copular density and tail independence. After that the paper presents a one-parameter family of distributions, and comments on Lévy copulas. Conclusions complete the discourse.

Illuminating the text is a battery of 17 figures and a series of numerical calculations.

For general discussions on dependence and related concepts see these references (Durbin and Stuart 1951; Schweizer and Wolff 1981; Genest, Ghoudi, and Rivest 1995; Joe 1997; Embrechts, McNeil, and Straumann 2002; Breymann, Dias, and Embrechts 2003; Lindskog, McNeil, and Schmock 2003; Nyfeler 2003; Tankov 2003; Patton 2004; Tankov 2004). Papers emphasizing tail dependence are these (Schmidt and Stadtmüller 2003; Schmidt 2005). For material more specific to copulas see these (Fréchet 1951; Sklar 1959; Genest and MacKay 1986; Genest and Rivest 1993; Joe 1993; Shih and Louis 1995; Sklar 1996; Nelsen 1998; Embrechts, McNeil, and Lindskog 2003; Carrière 2004; Cherubini, Luciano, and Vecchiato 2004; Cont and Tankov 2004).

2. MOTIVATION

As an exercise leading to this study, the author investigated a discrete distribution, the binomial pyramid, in two dimensions constructed from binomial coefficients of order n . This distribution has binomial margins, though is not bivariate binomial. In the limit as $n \rightarrow \infty$ the binomial pyramid frequency function $p_n(x, y)$, to be defined in Equation (2.2), induces a binomial distribution function which converges to the [continuous] pyramid distribution $G(x, y)$ introduced by this paper in Equation (3.4).

The pyramid density, provided in Equation (3.1), has the shape of the normal distribution function scaled by $1/2$ in each of its four axial branches. Though this continuous distribution stands on its own from its definition, the inspiration from the discrete model guided the further investigation and serves now as foundation.

As a preliminary step consider a domain D_n consisting of $(n + 1) \times (n + 1)$ points on an integer lattice centered at the origin of \mathbb{R}^2 . As n is even these points have integer coordinates; as n is odd the points have half integer coordinates. This convention in defining D_n allows for the consideration of both even and odd n at once.

The plan is to construct a function $P_n(x, y)$ with binomial margins on this lattice, with level sets on the nested squares of D_n . From $P_n(x, y)$ will follow the normalized frequency function $p_n(x, y)$, defined on a related lattice d_n . Within D_n are $\lceil \frac{n+1}{2} \rceil$ nested squares, with the center square having either a degenerate 1 point should n be even, or 4 points should n be odd. Thus the outermost square has $4n$ points. Within this square is another with $4(n - 2)$ points, *etc.*

Now let

$$I_n := \left\{ \frac{n}{2}, \frac{n}{2} - 1, \dots, \min \left(\frac{1}{2}, 0 \right) \right\}$$

be an index set. Then the domain of $P_n(x, y)$ conforms to

$$(2.1) \quad D_n := \{ (x, y) \mid (|x|, |y|) \in I_n \times I_n \}$$

Next, define $P_n(x, y)$ as the sum of binomial coefficients on D_n . To this end, let

$$L := \frac{n}{2} - \max(|x|, |y|), (x, y) \in D_n$$

Then

$$P_n(x, y) := \sum_{k=0}^L \binom{n+1}{k}$$

This last definition implies that $P_n(x, y) = \binom{n+1}{0}$ if either $|x| = n/2$, or $|y| = n/2$, *i.e.*, on points of the outer square of D_n . On the next included square of D_n , $P_n(x, y) = \binom{n+1}{0} + \binom{n+1}{1} = 1 + (n+1) = n+2$, *etc.*

A simple inductive argument establishes that the common margins of $P_n(x, y)$, defined as $P_n(\cdot)$, are the scaled binomial frequencies as here, assuming equiprobable events. The mean of this distribution is zero, the variance $\sigma_n^2 = n/4$, and therefore the standard deviation $\sigma_n = \sqrt{n}/2$.

$$P_n(x) = (n+1) \binom{n}{\frac{n}{2} - x},$$

so

$$P_n(y) = (n+1) \binom{n}{\frac{n}{2} - y}$$

The next task is to normalize $P_n(x, y)$ to $p_n(x, y)$. To start let

$$\widehat{I}_n := \frac{I_n}{\sigma_n},$$

where division by σ_n is implied for each component of I_n . Then in analogy to the definition for D_n in Equation (2.1), let

$$d_n := \left\{ (x, y) \mid (|x|, |y|) \in \widehat{I}_n \times \widehat{I}_n \right\} = \left\{ (x, y) \mid (\sigma|x|, \sigma|y|) \in I_n \times I_n \right\}$$

As above, consider d_n centered at the origin of \mathbb{R}^2 , and define

$$(2.2) \quad p_n(x, y) := \frac{P_n(\sigma_n x, \sigma_n y) \sigma_n^2}{(n+1) 2^n} = \frac{P_n(\frac{\sqrt{n}}{2} x, \frac{\sqrt{n}}{2} y)}{2^{n+2}} \binom{n}{\frac{n}{2} + x}$$

Then, $p_n(x, y)$ is a frequency function with common equiprobable binomial margins $p_n(\cdot)$. In particular, for $|x| \in \widehat{I}_n$, and $|y| \in \widehat{I}_n$,

$$p_n(x) = \sigma 2^{-n} \binom{n}{\frac{n}{2} + \sigma x} = \sqrt{n} 2^{-(n+1)} \binom{n}{\frac{n}{2} + \frac{\sqrt{n}}{2} x},$$

so

$$p_n(y) = \sigma 2^{-n} \binom{n}{\frac{n}{2} + \sigma y} = \sqrt{n} 2^{-(n+1)} \binom{n}{\frac{n}{2} + \frac{\sqrt{n}}{2} y}$$

Remark. Note that the bivariate binomial frequency function $p_n(x, y)$ has a total mass of σ_n^2 , and the common marginal binomial frequency function $p_n(\cdot)$ has a total mass of σ_n . The necessity of this scaling devolves from the fact that the lattice points on which $p_n(x, y)$ is defined are separated by $1/\sigma_n$ on each axis. Also, observe that the lower term of the binomial coefficient in the expression for $p_n(\cdot)$ is always an integer if n is even, and a half integer if n is odd. In the latter case the coefficient is calculated with reference to the gamma function. ■

Lastly, let

$$\begin{aligned} M &:= \left\lfloor \frac{n}{2} + \frac{\sqrt{n}}{2}x \right\rfloor \\ N &:= \left\lfloor \frac{n}{2} + \frac{\sqrt{n}}{2}y \right\rfloor \\ \Phi_n(x, y) &:= \frac{1}{\sigma_n^2} \sum_{x=0}^N \sum_{y=0}^M p_n(x, y) \\ \Phi_n(x) &:= \frac{1}{\sigma_n} \sum_{y=0}^M p_n(x, y) \\ \Phi_n(y) &:= \frac{1}{\sigma_n} \sum_{x=0}^N p_n(x, y) \end{aligned}$$

Then

$$\begin{aligned} \Phi_n(x, y) &\xrightarrow[n \rightarrow \infty]{\mathcal{D}} G(x, y) \\ \Phi_n(x) &\xrightarrow[n \rightarrow \infty]{\mathcal{D}} N_{0,1}(x) \\ \Phi_n(y) &\xrightarrow[n \rightarrow \infty]{\mathcal{D}} N_{0,1}(y) \end{aligned}$$

$N_{0,1}(\cdot)$ is the normal distribution function, and the convergences are in distribution, after extending $\Phi_n(x, y)$ and $\Phi_n(\cdot)$ to \mathbb{R}^2 and \mathbb{R} , respectively, by assigning probability zero to Borel sets which do not contain points of the lattices.

For reference, see Figures 1 and 2, which show $p_{16}(x, y)$ and its level curves.

3. THE PYRAMID

The pyramid distribution is an example of a two dimensional probability distribution that is normal on its margins, but is not bivariate normal. On the sample space $(x, y) \in \mathbb{R}^2$ it has a density not differentiable on the diagonals $y = \pm x$, and has tails thinner than the bivariate normal in all directions. As such, the distribution has an interesting copula, one that will figure in the sequel.

See Figures 3, 4, and 5, which show, in order, the density of the pyramid distribution, the level curves of this density, and a scatter plot of 2000 points taken from the distribution.

3.1. The density. The pyramid distribution is defined by its density. Given the normal density and distribution, respectively, as $f(x)$ and $F(x)$, the pyramid density is

$$(3.1) \quad g(x, y) := \frac{1}{2}F(x \wedge y \wedge -x \wedge -y) = \frac{1}{2}F(-(|x| \vee |y|))$$

In words, the pyramid density has the shape of the normal distribution function of the lower half line on each of its four faces. These faces come to a point at the origin. That the tails are thin derives from the fact that $F(x) = o[f(x)]$, as seen by a simple application of l'Hôpital's Rule.

From this definition, one calculates the distribution readily, the subject of Proposition 3.5 below. As well, this distribution is uncorrelated by Pearson's product-moment statistic, though the variables are dependent.

First, the task is to prove that $g(x, y)$ is a density, that is, that it integrates to 1, and that the marginal distributions are normal. Establishing the distribution follows. This lemma and corollary are useful.

Lemma 3.1.

$$I_1(x) := \int_{-\infty}^x F(y) \, dy = xF(x) + f(x)$$

$$I_2(x) := \int_{-\infty}^x yF(y) \, dy = \frac{1}{2} [(x^2 - 1)F(x) + xf(x)]$$

Proof. The two sides have the same derivative, and vanish at $-\infty$. □

Corollary 3.2.

$$I_1(0) = \int_{-\infty}^0 F(y) \, dy = \frac{1}{\sqrt{2\pi}}$$

$$I_2(0) = \int_{-\infty}^0 yF(y) \, dy = -\frac{1}{4}$$

Remark. By Corollary 3.2, $-4I_2(0) = 1$, and therefore $-4yF(y)$ is a density on the interval $(-\infty, 0]$; by Lemma 3.1, $-4I_2(x)$ is its distribution. This is the distribution of the pyramid in the sense of Lebesgue integration, wherein the differential of mass on the square $\{(x, y) \cup (y, x) \mid |x| \leq |y|\}$, taking y to be non-positive, is the perimeter $-8y$ times the uniform density on the square $\frac{1}{2}F(y)$. This distribution therefore provides the total mass outside the centered square with edges of length $-2x, x \leq 0$. Informally, this line of reasoning demonstrates that $g(x, y)$ is a density. The discussion proceeds now by conventional means. ■

Proposition 3.3. $g(x, y)$ is a density and the marginal distributions are normal.

Proof. First, choose $x \leq 0$. The value of the marginal density $g_1(x)$ is an integral through three sections of the joint density. Two of the sections, on the slopes in the positive and negative y axis directions, provide equal contributions to the marginal density. The third section, on the traverse of the slope in the direction of the negative x axis, provides a single

contribution. Therefore,

$$\begin{aligned}
 (3.2) \quad g_1(x) &= \int_{-\infty}^{\infty} g(x, y) \, dy \\
 &= 2 \int_{-\infty}^x \frac{1}{2} F(y) \, dy + \int_x^{-x} \frac{1}{2} F(x) \, dy \\
 &= \int_{-\infty}^x F(y) \, dy + \frac{1}{2} F(x)(-2x) \\
 &= f(x)
 \end{aligned}$$

by Lemma 3.1. For $x > 0$, symmetry in the pyramid density implies $g_1(x) = g_1(-x) = f(-x) = f(x)$.

The procedure for y is similar, as follows.

$$\begin{aligned}
 (3.3) \quad g_2(y) &= \int_{-\infty}^{\infty} g(x, y) \, dx \\
 &= 2 \int_{-\infty}^y \frac{1}{2} F(x) \, dx + \int_y^{-y} \frac{1}{2} F(y) \, dx \\
 &= \int_{-\infty}^y F(x) \, dx + \frac{1}{2} F(y)(-2y) \\
 &= f(y)
 \end{aligned}$$

by Lemma 3.1. For $y > 0$, symmetry in the pyramid density implies $g_2(y) = g_2(-y) = f(-y) = f(y)$.

As either marginal measure is a density, so is the bivariate measure, by Fubini's Theorem. \square

3.2. Correlation.

Proposition 3.4. *The pyramid distribution is uncorrelated, by Pearson's product-moment statistic.*

Proof. Proceed to compute the covariance. Owing to symmetry of $g(x, y)$,

$$\begin{aligned}
 \int_{-\infty}^0 \int_{-\infty}^0 xy \cdot g(x, y) \, dx \, dy &= \int_0^{\infty} \int_0^{\infty} xy \cdot g(x, y) \, dx \, dy \\
 &= - \int_0^{\infty} \int_{-\infty}^0 xy \cdot g(x, y) \, dx \, dy = - \int_{-\infty}^0 \int_0^{\infty} xy \cdot g(x, y) \, dx \, dy
 \end{aligned}$$

The sum of these four terms is $E[XY] = 0$. \square

See Figure 6, which shows the pyramid distribution function.

3.3. The distribution.

Proposition 3.5. *The pyramid distribution*

$$(3.4) \quad G(a, b) = \Pr\{X \leq a, Y \leq b\} \\ = \begin{cases} \frac{1}{2}F(a \wedge b)(ab + 1) + \frac{1}{2}(a \vee b)f(a \wedge b) & \text{if } a + b \leq 0 \\ G(-a, -b) + F(a) + F(b) - 1 & \text{if } a + b > 0 \end{cases}$$

Proof. First consider the case $a + b \leq 0$, $a \leq b$. Among the several approaches to this probability is the following.

$$(3.5) \quad G(a, b) = \Pr\{X \leq a, Y \leq b\} \\ = \int_{-\infty}^a \int_x^b \frac{1}{2}F(x) dy dx + \int_{-\infty}^a \int_y^a \frac{1}{2}F(y) dx dy \\ = \frac{1}{2}F(a)(ab + 1) + \frac{1}{2}bf(a)$$

This is a straightforward exercise in integration by parts, using Lemma 3.1. Similarly, in the case $a + b \leq 0$, $a > b$,

$$(3.6) \quad G(a, b) = \Pr\{X \leq a, Y \leq b\} \\ = \int_{-\infty}^b \int_y^a \frac{1}{2}F(y) dx dy + \int_{-\infty}^b \int_x^b \frac{1}{2}F(x) dy dx \\ = \frac{1}{2}F(b)(ab + 1) + \frac{1}{2}af(b)$$

Using the same methods and combining results, the assertion for this case follows. The result for the case $a + b > 0$ devolves from the fact that the marginal distributions are normal. \square

3.4. Kendall's tau. A commonly addressed non-parametric measure of a distribution's degree of association is Kendall's tau. This value is zero for the pyramid distribution, a consequence of the following lemma.

Lemma 3.6. *Kendall's tau τ_F for a distribution $F(x, y)$ is zero if the probability measure is invariant under rotation by $\pi/2$.*

Proof. From the definition,

$$\tau_F = \Pr\{(x_1 - x_2)(y_1 - y_2) > 0\} - \Pr\{(x_1 - x_2)(y_1 - y_2) < 0\}$$

A rotation of the measure by $\pi/2$ takes $(x_1, y_1) \rightarrow (-y_1, x_1)$ and $(x_2, y_2) \rightarrow (-y_2, x_2)$, reversing the concordance/discordance relationship as follows.

$$\begin{aligned} & \Pr\{(-y_1 + y_2)(x_1 - x_2) > 0\} - \Pr\{(-y_1 + y_2)(x_1 - x_2) < 0\} \\ &= [\Pr\{-(x_1 - x_2)(y_1 - y_2) > 0\} - \Pr\{-(x_1 - x_2)(y_1 - y_2) < 0\}] \\ &= -[\Pr\{(x_1 - x_2)(y_1 - y_2) > 0\} - \Pr\{(x_1 - x_2)(y_1 - y_2) < 0\}] \\ &= -\tau_F \end{aligned}$$

By the hypothesis, therefore, $\tau_F = -\tau_F$, and

$$\tau_F = 0$$

□

Corollary 3.7. $\tau_F = 0$, when $F(x, y)$ is the pyramid distribution.

Proof. On rotation, $f(x, y) = f(-y, x)$, satisfying the hypothesis of Lemma (3.6). □

4. A GENERAL TRANSFORMATION

The results of the previous section suggest a process, or transformation, which takes the normal density back to itself. The process looks like this with several functions leading to others.

$$f(x) \xrightarrow{\text{Dist}} F(x) \xrightarrow{\text{Cons}} g(x, y) \xrightarrow{\text{Dist}} G(a, b) \xrightarrow{\text{Marg}} g_1(x) = g_2(x) = f(x),$$

where the superscripts ‘Dist’, ‘Cons’, and ‘Marg’ stand respectively for the processes of computing the distribution from the density, constructing the pyramid density, and determining the marginal density. This process is a transformation T , with $f(x)$ as a fixed point, thus.

$$T : f(x) \mapsto f(x)$$

To give meaning to T for a more general class of densities it is necessary to be more specific about the construction step. In the present instance it was sufficient to define $g(x, y)$ as in Equation (3.1). In the general case it is necessary to guarantee that the construction produces a bivariate density, *i.e.*, that the resulting function integrates to 1 over \mathbb{R}^2 . This is simply a matter of scaling, for if Equation (3.1) had been expressed instead for a preliminary unnormalized $\bar{g}(c; x, y)$ with coefficient c instead of the coefficient $\frac{1}{2}$, as this,

$$(4.1) \quad \bar{g}(c; x, y) := cF(x \wedge y \wedge -x \wedge -y) = cF(-(|x| \vee |y|)),$$

then a coefficient $c_0 = \frac{1}{2}$ could have been determined readily as

$$c_0 = \left[\int_{\mathbb{R}^2} \bar{g}(1, x, y) \, dx \, dy \right]^{-1},$$

so that $g(x, y) = \bar{g}(c_0; x, y)$ is a density.

With this scaling any density is a candidate for applying the transformation T . A reasonable avenue of research, then, is to investigate T and its properties. Of particular interest is the set of fixed point densities, which would be those satisfying Equations (3.2) and (3.3) for a general $g(x, y)$. These are side issues for the present, but are revisited in Section 9 on conclusions below.

5. THE CHARACTERISTIC FUNCTION

Calculating the characteristic function is a straightforward task. Natural symmetries in the density suggest a piecewise approach, starting with a “West” quadrant, follow by axial symmetry to an “East” quadrant, followed by rotational symmetry to “South” and “North” quadrants. The first subsection defines this multiple domain and produces the function with corollary results. Following are brief comments on the distribution not having the infinite divisibility property, and on scaling the domain along the axes.

5.1. Calculation. The domain of the pyramid density naturally separates into four subdomains, those being the quadrants bounded by $y = \pm x$ in the plane on which boundaries the density is only of class C^0 (elsewhere being of class C^∞ .) For convenience call these quadrants, respectively, West, East, South, and North, or abbreviated, W, E, S, and N. Specifically these are

$$\begin{aligned} W &:= \{(x, y) \mid x < 0, x < y < -x\} \\ E &:= \{(x, y) \mid x > 0, -x < y < x\} \\ S &:= \{(x, y) \mid y < 0, y < x < -y\} \\ N &:= \{(x, y) \mid y > 0, -y < x < y\} \end{aligned}$$

Calculation of the characteristic function, stated formally as a proposition, first needs the support of a lemma and corollary.

Lemma 5.1. *If $a \neq 0$,*

$$\begin{aligned} \int_{-\infty}^x F(y) \sinh ay \, dy &= \frac{1}{a} \left[F(x) \cosh ax - \frac{1}{2} [F(x-a) + F(x+a)] \exp\left(\frac{a^2}{2}\right) \right] \\ \int_{-\infty}^x F(y) \cosh ay \, dy &= \frac{1}{a} \left[F(x) \sinh ax - \frac{1}{2} [F(x-a) - F(x+a)] \exp\left(\frac{a^2}{2}\right) \right] \end{aligned}$$

Proof. In each case, the two sides have the same derivative, which vanishes at $-\infty$. In the second case establishing the result requires reliance on the fact that $F(x)$ is $o[\exp(-ax)]$ as $x \rightarrow -\infty$, a direct conclusion following an application of l'Hôpital's Rule. \square

Corollary 5.2. *If $a \neq 0$,*

$$\begin{aligned} \int_{-\infty}^0 F(y) \sinh ay \, dy &= \frac{1}{2a} \left[1 - \exp\left(\frac{a^2}{2}\right) \right] \\ \int_{-\infty}^0 F(y) \cosh ay \, dy &= \frac{1}{2a} [2F(a) - 1] \exp\left(\frac{a^2}{2}\right) \end{aligned}$$

Proposition 5.3. *The pyramid distribution has characteristic function*

$$\varphi(\zeta, \eta) = \frac{1}{\zeta\eta} \sinh(\zeta\eta) \gamma(\zeta, \eta),$$

where

$$\gamma(\zeta, \eta) := \exp\left(-\frac{\zeta^2 + \eta^2}{2}\right)$$

is the characteristic function of the independent binormal distribution.

Proof. Insofar as the density is symmetric among these quadrants about the origin, one may develop the characteristic function by integrating over one of the quadrants, then exploit this symmetry to finish the calculation. The choice is arbitrary, so begin with the West subdomain.

Let $\varphi_W(\zeta, \eta)$ be the characteristic function restricted to the West subdomain, with parallel definitions, $\varphi_E(\zeta, \eta)$, $\varphi_S(\zeta, \eta)$, $\varphi_N(\zeta, \eta)$, for the other subdomains. Then,

$$(5.1) \quad \varphi_W(\zeta, \eta) = \int_{-\infty}^0 \int_x^{-x} e^{i(\zeta x + \eta y)} \cdot \frac{1}{2} F(x) \, dy \, dx$$

$$(5.2) \quad = \frac{1}{2} \int_{-\infty}^0 e^{i\zeta x} F(x) \int_x^{-x} e^{i\eta y} \, dy \, dx$$

$$(5.3) \quad = \frac{i}{\eta} \int_{-\infty}^0 e^{i\zeta x} F(x) \sinh i\eta x \, dx$$

Similarly, $\varphi_E(\zeta, \eta)$ has a definition

$$(5.4) \quad \varphi_E(\zeta, \eta) = \int_0^{\infty} \int_{-x}^x e^{i(\zeta x + \eta y)} \cdot \frac{1}{2} [1 - F(x)] \, dy \, dx$$

Now, to get $\varphi_E(\zeta, \eta)$ into a form analogous to $\varphi_W(\zeta, \eta)$ do the following. Change the variable x to $-x$, recognize that $[1 - F(x)] = F(-x)$, and exchange the outer limits of integration. Arrive at the following expression, which differs from $\varphi_W(\zeta, \eta)$ of Equation (5.2) only on the sign of the first exponent.

$$(5.5) \quad \varphi_E(\zeta, \eta) = \frac{1}{2} \int_{-\infty}^0 e^{-i\zeta x} F(x) \int_x^{-x} e^{i\eta y} \, dy \, dx$$

Continuing, as above for $\varphi_W(\zeta, \eta)$, calculate that

$$(5.6) \quad \varphi_E(\zeta, \eta) = \frac{i}{\eta} \int_{-\infty}^0 e^{-i\zeta x} F(x) \sinh i\eta x \, dx$$

This process prepares one to add the results for $\varphi_W(\zeta, \eta)$ and $\varphi_E(\zeta, \eta)$ of Equations (5.3) and (5.6) to get a combined result for the characteristic function including both the West and

East subdomains. Call this sum $\varphi_{WE}(\zeta, \eta)$. Then, as one readily computes,

$$(5.7) \quad \begin{aligned} \varphi_{WE}(\zeta, \eta) &= \varphi_W(\zeta, \eta) + \varphi_E(\zeta, \eta) \\ &= \frac{2i}{\eta} \int_{-\infty}^0 F(x) \cosh i\zeta x \sinh i\eta x \, dx \end{aligned}$$

The pyramid density $g(x, y)$ is symmetric in its variables. Therefore, the analogous combined characteristic function including both the South and North subdomains, $\varphi_{SN}(\zeta, \eta)$, is achieved by interchanging the roles, respectively, of x and y , and of ζ and η .

$$(5.8) \quad \varphi_{SN}(\zeta, \eta) = \frac{2i}{\zeta} \int_{-\infty}^0 F(y) \cosh i\eta y \sinh i\zeta y \, dy$$

Next, convert the last expressions for $\varphi_{WE}(\zeta, \eta)$ and $\varphi_{SN}(\zeta, \eta)$, Equations (5.7) and (5.8), respectively, using the identity

$$\cosh ax \sinh bx = \frac{1}{2} [\sinh(a+b)x - \sinh(a-b)x]$$

At the same time, recognize that the variables of integration are simply formal, and change y to x in the second expression. As well, negate the argument in the final hyperbolic sine, so as to conform it to the argument above it, while reversing the sign on that term to maintain the expression.

$$\begin{aligned} \varphi_{WE}(\zeta, \eta) &= \frac{i}{\eta} \int_{-\infty}^0 F(x) [\sinh i(\zeta + \eta)x - \sinh i(\zeta - \eta)x] \, dx \\ \varphi_{SN}(\zeta, \eta) &= \frac{i}{\zeta} \int_{-\infty}^0 F(x) [\sinh i(\zeta + \eta)x + \sinh i(\zeta - \eta)x] \, dx \end{aligned}$$

One can resolve these integrations by recourse to the first part of Corollary 5.2.

$$\begin{aligned} \varphi_{WE}(\zeta, \eta) &= \frac{1}{2\eta(\zeta + \eta)} \left[1 - \exp\left(-\frac{(\zeta + \eta)^2}{2}\right) \right] - \frac{1}{2\eta(\zeta - \eta)} \left[1 - \exp\left(-\frac{(\zeta - \eta)^2}{2}\right) \right] \\ \varphi_{SN}(\zeta, \eta) &= \frac{1}{2\zeta(\zeta + \eta)} \left[1 - \exp\left(-\frac{(\zeta + \eta)^2}{2}\right) \right] + \frac{1}{2\zeta(\zeta - \eta)} \left[1 - \exp\left(-\frac{(\zeta - \eta)^2}{2}\right) \right] \end{aligned}$$

A few steps now of collecting terms (vertically on the first factors) provides the complete characteristic function $\varphi(\zeta, \eta) = \varphi_{WE}(\zeta, \eta) + \varphi_{SN}(\zeta, \eta)$, as follows.

$$\begin{aligned} \varphi(\zeta, \eta) &= \frac{1}{2\zeta\eta} \left[\exp\left(-\frac{(\zeta - \eta)^2}{2}\right) - \exp\left(-\frac{(\zeta + \eta)^2}{2}\right) \right] \\ &= \frac{1}{\zeta\eta} \sinh(\zeta\eta) \exp\left(-\frac{\zeta^2 + \eta^2}{2}\right) \\ &= \frac{1}{\zeta\eta} \sinh(\zeta\eta) \gamma(\zeta, \eta) \end{aligned}$$

□

Corollary 5.4.

$$\varphi(\zeta, \eta) = \frac{1}{2} \int_{-1}^{+1} \exp(\zeta \eta \cdot x) dx \cdot \gamma(\zeta, \eta)$$

Corollary 5.5. *The marginal distributions are normal.*

$$\varphi(\zeta, 0) = \varphi(0, \zeta) = \exp\left(-\frac{\zeta^2}{2}\right) =: \gamma(\zeta)$$

Remark. In the development thus far, the theory has focused attention only on the bivariate normal distribution in the independent case. In this regard one could note that $\gamma(\zeta, \eta) = \gamma(\zeta)\gamma(\eta)$ and incorporate this fact into the analysis. ■

See Figures 7, 8, 9, and 10, which show, in order, the characteristic function of the pyramid distribution, its level curves, the characteristic function of the normal distribution, and its level curves.

The difference of the pyramid and normal characteristic functions,

$$\omega(\zeta, \eta) := \varphi(\zeta, \eta) - \gamma(\zeta, \eta),$$

has maxima symmetrically placed in the four quadrants. The maxima, determined numerically, occur at

$$(\zeta, \eta) = (\pm 1.651901, \pm 1.651901)$$

The value at those points is 0.117152.

See Figures 11 and 12, which show the difference of the pyramid and normal characteristic functions, and the level curves of this difference.

5.2. Infinite divisibility. The question of infinite divisibility arises in the context of Lévy processes. In particular, could the pyramid distribution be infinitely divisible, and accordingly be the basis for defining directly a Lévy process with associated Lévy measure? The following proposition answers the question in the negative. However, see below to Section 8 for a discussion of the induction of Lévy copulas related to the pyramid distribution from its ordinary copula. These Lévy copulas create bivariate Lévy measures from marginal Lévy measures, thereby allowing the construction of bivariate Lévy processes. The laws of all Lévy processes are infinitely divisible. For an excellent treatment of infinite divisibility and related properties, see this (Itô 1942).

Proposition 5.6. *The pyramid distribution is not infinitely divisible.*

Proof. A result of Sato implies this conclusion (Sato 1999, PROPOSITION 11.10, p. 65). Specifically, a Lévy process with non-trivial Lévy measure must have non-trivial Lévy measure on at least one of the projected processes. Thus for the pyramid distribution to be infinitely divisible it must be purely Gaussian, which it is not. □

5.3. Scalability. The Pyramid distribution is scalable, with density $\hat{g}(x, y)$ as follows.

$$\hat{g}(x, y) = c^2 g\left(cx, \frac{c}{a}y\right)$$

where $g(x, y)$ is the Pyramid density, $a > 0$, $c > 0$. All the results of this paper apply appropriately to the scaled distribution, except for symmetry if $a \neq 1$.

6. THE COPULA

Let $\alpha = F(a), \beta = F(b)$. Then the copula for any distribution $D(a, b)$ with margins $D_1(a)$ and $D_2(b)$ is a function

$$H: [0, 1] \times [0, 1] \rightarrow [0, 1]$$

$$H(D_1(a), D_2(b)) = H(\alpha, \beta) = D(D_1^{-1}(\alpha), D_2^{-1}(\beta)) = D(a, b)$$

For $G(a, b)$ with margins $F(a)$ and $F(b)$ this copula becomes

$$H(F(a), F(b)) = H(\alpha, \beta) = G(F^{-1}(\alpha), F^{-1}(\beta)) = G(a, b)$$

See Figure 13, which shows the copula of the pyramid distribution.

Following the calculation of the specific functional form for the copula, this section proceeds to calculate Spearman's rho and Blomqvist's beta, produces the copular density, and discusses tail independence.

6.1. Specific functional form. $H(\alpha, \beta)$ has a specific form implied by $G(a, b)$ given by Proposition 3.5. First needed is this lemma.

Lemma 6.1. $\alpha + \beta \leq 1 \iff a + b \leq 0$. Also, $\alpha \leq \beta \iff a \leq b$.

Proof.

$$\alpha + \beta \leq 1 \iff \alpha \leq 1 - \beta \iff F^{-1}(\alpha) \leq F^{-1}(1 - \beta) \iff a \leq -b \iff a + b \leq 0$$

$$\alpha \leq \beta \iff F^{-1}(\alpha) \leq F^{-1}(\beta) \iff a \leq b \quad \square$$

Then,

$$(6.1) \quad H(\alpha, \beta) = G(F^{-1}(\alpha), F^{-1}(\beta))$$

$$= \begin{cases} \frac{1}{2}(\alpha \wedge \beta) (F^{-1}(\alpha)F^{-1}(\beta) + 1) + \frac{1}{2} (F^{-1}(\alpha) \vee F^{-1}(\beta)) f(F^{-1}(\alpha) \wedge F^{-1}(\beta)) & \text{if } \alpha + \beta \leq 1 \\ G(-F^{-1}(\alpha), -F^{-1}(\beta)) + \alpha + \beta - 1 & \text{if } \alpha + \beta > 1 \end{cases}$$

One may compare this copula with that of the independent copula

$$(6.2) \quad C_{\perp}: [0, 1] \times [0, 1] \rightarrow [0, 1]$$

$$C_{\perp}(\alpha, \beta) = \alpha\beta$$

See Figure 14, which shows the product copula, which represents all independent distributions.

Further, one may look to the difference function

$$(6.3) \quad \Delta_H: [0, 1] \times [0, 1] \rightarrow [0, 1]$$

$$\Delta_H(\alpha, \beta) := H(\alpha, \beta) - C_{\perp}(\alpha, \beta)$$

See Figures 15 and 16, which show the difference of the pyramid and independent copulas, and the level curves of this difference.

This copula difference exhibits a property inspiring a definition, and then two propositions.

Definition 6.2. A bivariate distribution with copula $K(\alpha, \beta)$ is *symmetrically dependent* if its copula difference to the independent copula $K_{\Delta}(\alpha, \beta) := K(\alpha, \beta) - C_{\perp}(\alpha, \beta)$ satisfies the following conditions:

(1)

$$\begin{aligned}\Delta_K(\alpha, \beta) &= \Delta_K(1 - \alpha, 1 - \beta) = -\Delta_K(1 - \alpha, \beta) = -\Delta_K(\alpha, 1 - \beta), \\ \forall \{\alpha, \beta\} &\in [0, 1] \times [0, 1]\end{aligned}$$

(2) $\Delta_K(\alpha, \beta)$ is not identically zero (the case of independence.)**Proposition 6.3.** *For a symmetrically dependent distribution with copula difference $\Delta_K(\alpha, \beta)$*

$$\int_0^1 \int_0^1 \Delta_K(\alpha, \beta) \, d\alpha \, d\beta = 0$$

Proof. From Definition 6.2

$$\begin{aligned}\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \Delta_K(\alpha, \beta) \, d\alpha \, d\beta &= \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \Delta_K(\alpha, \beta) \, d\alpha \, d\beta \\ &= - \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} \Delta_K(\alpha, \beta) \, d\alpha \, d\beta = - \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \Delta_K(\alpha, \beta) \, d\alpha \, d\beta,\end{aligned}$$

whence the conclusion follows. □**Proposition 6.4.** *The pyramid distribution is symmetrically dependent.**Proof.* PART 1 OF 3 —First establish that $\Delta_H(\alpha, \beta) = \Delta_H(1 - \alpha, 1 - \beta)$. Without loss of generality assume $\alpha + \beta > 1$. By Lemma 6.1, $a + b > 0$.

$$\begin{aligned}\Delta_H(\alpha, \beta) &= G[F^{-1}(\alpha), F^{-1}(\beta)] - C_{\perp}(\alpha, \beta) \\ &= G(a, b) - F(a)F(b) \\ &= [G(-a, -b) + F(a) + F(b) - 1] - F(a)F(b) \\ &= \left\{ \frac{1}{2} [((-a)(-b) + 1)F(-b) - bf(-a)] + [1 - F(-a)] + [1 - F(-b)] - 1 \right\} \\ &\quad - [1 - F(-a)][1 - F(-b)], \text{ by Proposition 3.5,} \\ &= \frac{1}{2} [((-a)(-b) + 1)F(-b) - bf(-a)] - F(-a)F(-b) \\ &= G[F^{-1}(1 - \alpha), F^{-1}(1 - \beta)] - C_{\perp}(1 - \alpha, 1 - \beta) \\ &= \Delta_H(1 - \alpha, 1 - \beta)\end{aligned}$$

PART 2 OF 3 —

Next, establish that $\Delta_H(\alpha, \beta) = -\Delta_H(\alpha, 1 - \beta)$. Examine two exhaustive cases, first for $\alpha \leq \beta$. By Lemma 6.1, $a \leq b$.

Case I: $\alpha + \beta \leq 1$. By Lemma 6.1, $a + b \leq 0$.

$$\begin{aligned}
 \Delta_H [F^{-1}(\alpha), F^{-1}(\beta)] &= G(a, b) - F(a)F(b) \\
 &= \frac{1}{2} [(ab + 1)F(a) + bf(a)] - F(a)F(b), \text{ by Proposition 3.5,} \\
 &= -\frac{1}{2} [(a(-b) + 1)F(a) - bf(a)] + F(a)[1 - F(b)] \\
 &= -[G(a, -b) - F(a)F(-b)] \\
 &= -\Delta_H [F^{-1}(\alpha), F^{-1}(1 - \beta)]
 \end{aligned}$$

Case II: $\alpha + \beta > 1$. By Lemma 6.1, $a + b > 0$.

$$\begin{aligned}
 \Delta_H [F^{-1}(\alpha), F^{-1}(\beta)] &= [G(-a, -b) + F(a) + F(b) - 1] - F(a)F(b) \\
 &= \left\{ \frac{1}{2} [((-a)(-b) + 1)F(-b) + (-a)f(-b)] + F(a) + [1 - F(-b)] - 1 \right\} \\
 &\quad - F(a)[1 - F(-b)], \text{ by Proposition 3.5,} \\
 &= \left\{ -\frac{1}{2} [(a(-b) - 1)F(-b) + af(-b)] + F(a) + [1 - F(-b)] - 1 \right\} \\
 &\quad - F(a) + F(a)F(-b) \\
 &= -\frac{1}{2} [(a(-b) + 1)F(-b) + af(-b)] + F(a)F(-b) \\
 &= -[G(a, -b) - F(a)F(-b)] \\
 &= -\Delta_H [F^{-1}(\alpha), F^{-1}(1 - \beta)]
 \end{aligned}$$

The two cases for $\alpha > \beta$ are analogous (interchanging α and β), establishing this Part.

PART 3 OF 3 —

Finally, by PART 1,

$$-\Delta_H(\alpha, 1 - \beta) = -\Delta_H(1 - \alpha, \beta),$$

and by PART 2,

$$-\Delta_H(1 - \alpha, \beta) = \Delta_H(1 - \alpha, 1 - \beta)$$

□

The copula difference $\Delta_H(\alpha, \beta)$

has four extrema, two maxima and two minima. Standard development to identify them leads to an analytically intractable equation; however, numeric solutions are available.

The maxima occur at

$$\begin{aligned}
 (\alpha, \alpha) &= (0.198089, 0.198089) \\
 (1 - \alpha, 1 - \alpha) &= (0.801911, 0.198089),
 \end{aligned}$$

and the minima occur at

$$\begin{aligned}
 (\alpha, 1 - \alpha) &= (0.198089, 0.801911) \\
 (1 - \alpha, \alpha) &= (0.801911, 0.198089)
 \end{aligned}$$

The respective maxima and minima are ± 0.01302284 .

6.2. Spearman's rho and Blomqvist's beta. Symmetric dependence, as exhibited in the pyramid distribution by Proposition (6.4) implies the Spearman's rho, a measure of rank correlation, is zero. That this is true is a consequence of a more general result, that Spearman's rho for any distribution is a simple scaling of the integral of the copula difference over its domain. This result is known and was stated by Nelsen, but not fully developed by him, omitting a few steps included now (Nelsen 1998, Theorem 5.1.6., page 135, and Equation (5.1.16), p. 138).

Proposition 6.5. *Spearman's rho ϱ_K for a distribution with copula $K(\alpha, \beta)$ and copula difference to the independent copula $K_\Delta(\alpha, \beta) := K(\alpha, \beta) - C_\perp(\alpha, \beta)$ is*

$$\varrho_K = 12 \int_0^1 \int_0^1 \Delta_K(\alpha, \beta) \, d\alpha \, d\beta$$

Proof.

$$\varrho_K = 12 \int_0^1 \int_0^1 K(\alpha, \beta) \, d\alpha \, d\beta - 3$$

From this it follows by Equation (6.2) that

$$\begin{aligned} \varrho_{C_\perp} &= 12 \int_0^1 \int_0^1 C_\perp(\alpha, \beta) \, d\alpha \, d\beta - 3 \\ &= 12 \int_0^1 \int_0^1 \alpha\beta \, d\alpha \, d\beta - 3 = 0 \end{aligned}$$

But then,

$$\begin{aligned} \varrho_K &= \varrho_K - \varrho_{C_\perp} = 12 \int_0^1 \int_0^1 [K(\alpha, \beta) - C_\perp(\alpha, \beta)] \, d\alpha \, d\beta \\ &= 12 \int_0^1 \int_0^1 K_\Delta(\alpha, \beta) \, d\alpha \, d\beta \quad \text{as asserted} \end{aligned}$$

□

Remark. This theorem provides the insight to visualize Spearman's rho by looking at the copula difference function and evaluating its integral. In addition, the theorem provides a basis for selecting a copula, and thereafter a distribution, with a desired value for Spearman's rho. ■

Corollary 6.6. *Spearman's rho for a symmetrically dependent distribution is zero.*

Corollary 6.7. *Spearman's rho ϱ_H for the pyramid distribution is zero.*

Remark. Of interest also are the Fréchet lower and upper limit copulas $C_\downarrow(v, z)$ and $C_\uparrow(v, z)$, respectively, representing complete negative and complete positive dependence. See (Cherubini, Luciano, and Vecchiato 2004, pp. 52–56) for details. One easily calculates Spearman's

rho for these copulas by the method of Proposition 6.5. As expected, $\varrho_K = \mp 1$.

$$\begin{aligned} C_{\downarrow}(v, z) &:= \max(v + z - 1, 0) && \text{the Fréchet lower bound copula} \\ \varrho_{C_{\downarrow}} &= -1 \\ C_{\uparrow}(v, z) &:= \min(v, z) && \text{the Fréchet upper bound copula} \\ \varrho_{C_{\uparrow}} &= +1 \end{aligned}$$

■

Blomqvist's beta is a valuation of a copula at it's center, scaled so that $\beta(C_{\downarrow}(v, z)) = -1$ and $\beta(C_{\uparrow}(v, z)) = +1$. As well, $\beta(C_{\perp}(v, z)) = 0$. Specifically, for any copula $C(v, z)$,

$$\beta = 4 \cdot C\left(\frac{1}{2}, \frac{1}{2}\right) - 1$$

In the present instance

$$\beta = 4 \cdot G\left(\frac{1}{2}, \frac{1}{2}\right) - 1 = 4 \cdot \frac{1}{4} - 1 = 0,$$

which follows by reference to Equation (3.5).

6.3. The copular density. One readily computes the density of the pyramid copula from the copula. If $h(\alpha, \beta)$ be this density, then

$$h(\alpha, \beta) = \frac{\partial^2}{\partial \alpha \partial \beta} \int_0^{\beta} \int_0^{\alpha} h(u, v) \, du \, dv = \frac{\partial^2}{\partial \alpha \partial \beta} H(\alpha, \beta)$$

Applying the chain rule to Equation (6.1) gives

$$h(\alpha, \beta) = \frac{g(F^{-1}(\alpha), F^{-1}(\beta))}{f(F^{-1}(\alpha))f(F^{-1}(\beta))}$$

Substituting the alternate pyramid density definition of Equation (3.1), one has

$$h(\alpha, \beta) = \frac{\frac{1}{2}[\alpha \wedge \beta \wedge (1 - \alpha) \wedge (1 - \beta)]}{f(F^{-1}(\alpha))f(F^{-1}(\beta))},$$

or

$$h(\alpha, \beta) = \frac{\frac{1}{2} \left[\frac{1}{2} - (|\alpha - \frac{1}{2}| \vee |\beta - \frac{1}{2}|) \right]}{f(F^{-1}(\alpha))f(F^{-1}(\beta))}$$

6.4. Tail dependence. The tails of the pyramid distribution are independent by the ordinary definition. Here it is, tailored to the present circumstances, followed by a Proposition.

Definition 6.8. A bivariate distribution is *lower tail dependent* with coefficient λ_L , $0 \leq \lambda_L \leq 1$ if

$$\lim_{a \rightarrow -\infty} \Pr \{Y \leq a \mid X \leq a\} = \lim_{a \rightarrow -\infty} \frac{G(a, a)}{F(a)} = \lim_{\alpha \rightarrow 0} \frac{H(\alpha, \alpha)}{\alpha} = \lambda_L$$

A bivariate distribution is *upper tail dependent* with coefficient λ_U , $0 \leq \lambda_U \leq 1$ if the distribution of $(-X, -Y)$ is lower tail dependent with coefficient λ_U . A distribution is either *lower tail independent* or *upper tail independent*, respectively, as $\lambda_L = 0$ or $\lambda_U = 0$. A distribution is either *lower tail completely dependent* or *upper tail completely dependent*, respectively, as $\lambda_L = 1$ or $\lambda_U = 1$.

Proposition 6.9. *The pyramid distribution is both lower and upper tail independent.*

Proof. By Proposition 3.5

$$\lambda_L = \lim_{a \rightarrow -\infty} \frac{G(a, a)}{F(a)} = \lim_{a \rightarrow -\infty} \frac{1}{2} \left[(a^2 + 1) + \frac{af(a)}{F(a)} \right] = 0$$

Applying l'Hôpital's Rule to this expression yields the result for the lower tail. The result for the upper tail follows by symmetry. \square

7. A ONE-PARAMETER FAMILY

One may define a family of distributions based on linear combinations of the pyramid and binormal densities, thus.

$$g_\theta(x, y) := \theta g(x, y) + (1 - \theta) f(x, y), \quad 0 \leq \theta \leq 1$$

Then this scaling propagates through the distribution function, characteristic function, and copula, owing to the linearity of the integral operator. For completeness here are the formalities, giving rise to a few definitions along the way. The functions $f(\cdot, \cdot)$ and $F(\cdot, \cdot)$ are the bivariate normal density and distribution, respectively.

Proposition 7.1.

The distribution function

$$G_\theta(a, b) := \int_{-\infty}^a \int_{-\infty}^b g_\theta(x, y) \, dy \, dx = \theta G(a, b) + (1 - \theta) F(a, b)$$

The characteristic function

$$\varphi_\theta(\zeta, \eta) := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\zeta x + \eta y)} g_\theta(x, y) \, dy \, dx = \theta \varphi(\zeta, \eta) + (1 - \theta) \gamma(\zeta, \eta)$$

The copula

$$\begin{aligned} H_\theta(\alpha, \beta) &:= G_\theta(F^{-1}(\alpha), F^{-1}(\beta)) = \theta H(\alpha, \beta) + (1 - \theta) C_\perp(\alpha, \beta) \\ &= C_\perp(\alpha, \beta) + \theta \Delta_H(\alpha, \beta) \end{aligned}$$

Proof.

The distribution function

$$\begin{aligned}
 G_\theta(a, b) &= \int_{-\infty}^a \int_{-\infty}^b g_\theta(x, y) \, dy \, dx \\
 &= \theta \int_{-\infty}^a \int_{-\infty}^b g(x, y) \, dy \, dx + (1 - \theta) \int_{-\infty}^a \int_{-\infty}^b f(x, y) \, dy \, dx \\
 &= \theta G(a, b) + (1 - \theta) F(a, b)
 \end{aligned}$$

The characteristic function

$$\begin{aligned}
 \varphi_\theta(\zeta, \eta) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\zeta x + \eta y)} g_\theta(x, y) \, dy \, dx \\
 &= \theta \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\zeta x + \eta y)} g(x, y) \, dy \, dx + (1 - \theta) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\zeta x + \eta y)} f(x, y) \, dy \, dx \\
 &= \theta \varphi(\zeta, \eta) + (1 - \theta) \gamma(\zeta, \eta)
 \end{aligned}$$

The copula

$$\begin{aligned}
 H_\theta(\alpha, \beta) &= G_\theta(F^{-1}(\alpha), F^{-1}(\beta)) \\
 &= \theta G(F^{-1}(\alpha), F^{-1}(\beta)) + (1 - \theta) F(F^{-1}(\alpha), F^{-1}(\beta)) \quad \text{as above} \\
 &= \theta H(\alpha, \beta) + (1 - \theta) C_\perp(\alpha, \beta) \\
 &= C_\perp(\alpha, \beta) + \theta \Delta_H(\alpha, \beta) \quad \text{by Equation (6.3)}
 \end{aligned}$$

□

Remark. Note that the $\varphi(\zeta, \eta)$ as calculated in Proposition 7.1 above is not the characteristic function of the sum of two random variables. Rather, it is the characteristic function of a single random variable defined by a density which is the convex combination of two other densities. As such, no questions of dependence arise.

Also, observe that the family of densities has fixed points placed symmetrically on the axes at $(x, 0) = (\pm 0.6510, 0)$ and $(0, y) = (0, \pm 0.6510)$, where $g_\theta(x, y) = 0.1288$. To see, take $\frac{\partial}{\partial \theta} g_\theta(x, y) = 0$ and solve.

As well, one could look toward extending the family to negative values of the parameter θ . This is not possible, for in such circumstances the density $g_\theta(x, y)$ would be negative in a neighborhood of a point on each of the two tails of the major diagonal. To see this, locate the minima of $\tilde{g}_\theta(x) := g_\theta(x, x)$. These are the points $\{x \mid -|x| f(x) = \theta / (4(1 - \theta))\}$. This equation has finite solutions for any $\theta < 0$. Furthermore, $\tilde{g}_\theta(x) < 0$ at these points as no stationary points more remote exist, and $\lim_{x \rightarrow \pm\infty} \tilde{g}_\theta(x) = 0$. The conclusion readily follows.

A similar statement obtains for minima on the minor diagonal, but the analysis above suffices. ■

See Figure 17. This figure shows values of the density in the axial directions $(x = 0) \vee (y = 0)$ for choices of θ . The fixed point appears.

Lastly, the entire one-parameter family is tail independent. This is easily conjectured, for the pyramid distribution is tail independent by Proposition 6.9, and the multivariate normal distribution is known also to be. Formally, this is the statement, following a lemma.

Lemma 7.2.

$$\begin{aligned}\frac{d}{da} \int_{-\infty}^a \int_{-\infty}^a f(x, y) dy dx &= 2F(a)f(a) = \frac{d}{da} F^2(a) \\ \frac{d}{da} \int_{-\infty}^a \int_{-\infty}^0 f(x, y) dy dx &= \frac{1}{2}f(a)\end{aligned}$$

Proof. Standard calculus methods establish the results. □

Proposition 7.3. $G_\theta(x, y)$ is both lower and upper tail independent.

Proof. Let $\lambda_L^{(\theta)}$ be the lower tail dependency coefficient. Then,

$$\lambda_L^{(\theta)} = \lim_{a \rightarrow -\infty} \frac{\theta G(a, a) + (1 - \theta)F(a, a)}{\theta F(a) + (1 - \theta)F(a)} = \lim_{a \rightarrow -\infty} \frac{\theta G(a, a) + (1 - \theta)F(a, a)}{F(a)} = 0$$

Applying l'Hôpital's Rule and Lemma 7.2 to this expression yields the result for the lower tail. The result for the upper tail follows by symmetry. □

Remark. Note that Proposition 7.3 proves Proposition 6.9 again as a special case (for $\theta = 1$) and also proves tail independence for the uncorrelated bivariate normal distribution (for $\theta = 0$). ■

8. LÉVY COPULAS

Tankov in his Ph.D. thesis provided a result which enables one to construct a Lévy copula from an ordinary (probability) copula (Tankov 2004, Theorem 5.1). Applying his result here states that for the pyramid copula $H(\alpha, \beta)$, the induced function

$$\begin{aligned}L : [0, \infty]^d &\rightarrow [0, \infty] \\ L(\gamma, \delta) &:= \psi \left(H(\psi^{-1}(\gamma), \psi^{-1}(\delta)) \right),\end{aligned}$$

for a strictly increasing function

$$\psi : [0, 1] \rightarrow [0, \infty]$$

having positive derivatives to order d on $(0, 1)$, is a Lévy copula.

Tankov offers as an example the function $\psi(x) = \frac{x}{1-x}$.

The Lévy copula is an important construct in defining and interpreting dependence relationships between and among Lévy processes. In this regard, the pyramid-derived Lévy copulas have interesting implications. The cited thesis, in addition to these works, is a good starting point to explore such ideas (Cont and Tankov 2004, Chapter 5)(Tankov 2003).

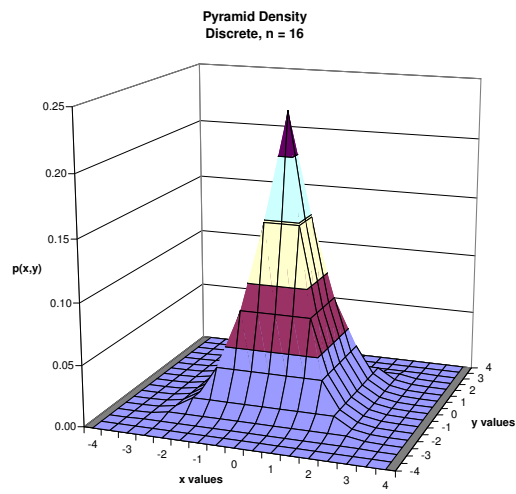
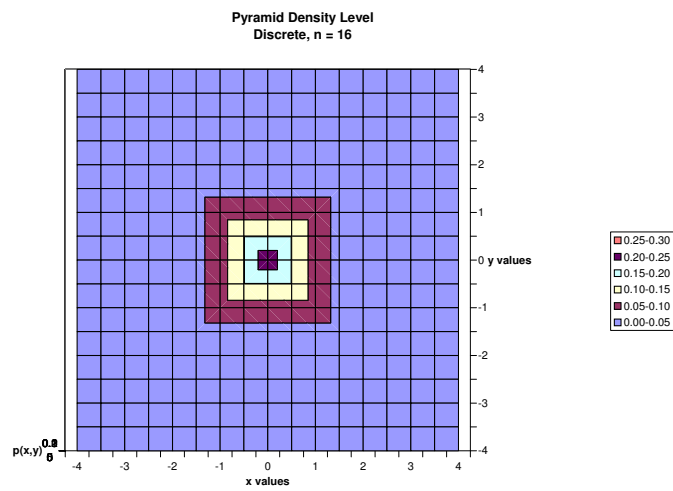
The concept carries forward to a Lévy copula for any member of the parametric family. With assumptions as above the function

$$L_{\theta}(\gamma, \delta) := \psi\left(H_{\theta}(\psi^{-1}(\gamma), \psi^{-1}(\delta))\right)$$

is also a Lévy copula. The proof is omitted.

9. CONCLUSIONS

The pyramid distribution stimulates interest in several threads of research. One is to investigate further the class of distributions with normal margins, made much more feasible by recent advances in copula theory. Another is to research distributions as building blocks for other distributions, generating margins which may have relations to the original blocks, as in the discussion of Section 4 on the general transformation T . A third is to seek applications for distributions with normal margins or other margins of interest, and for distributions constructed like architecture. Seen is a synergy between applications suggesting constructions, and constructions stimulating applications, including in finance, physics, and game theory, among others.

FIGURE 1. Pyramid Distribution Density, Discrete $n = 16$ FIGURE 2. Pyramid Distribution Density Level Curves, Discrete $n = 16$

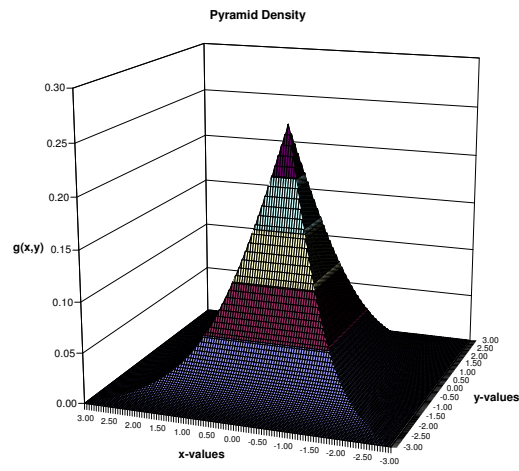


FIGURE 3. Pyramid Distribution Density

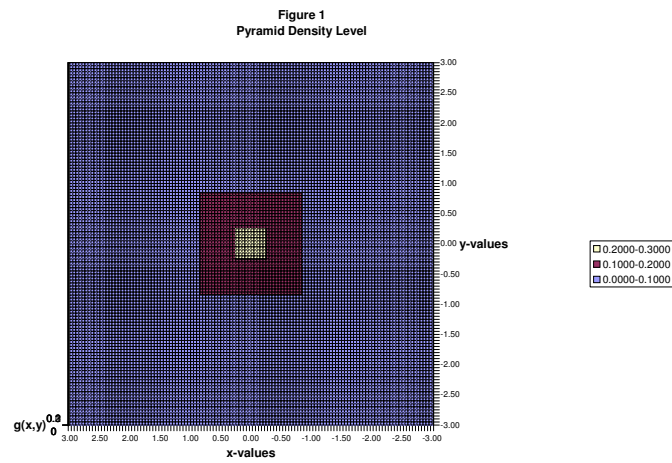


FIGURE 4. Pyramid Distribution Density Level Curves

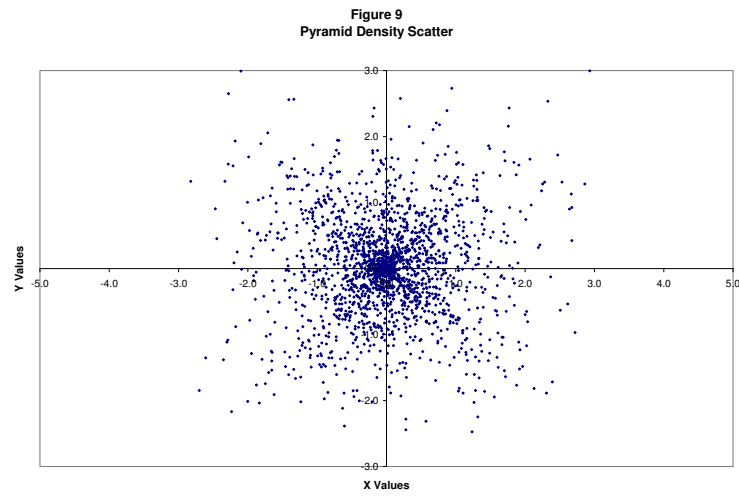


FIGURE 5. Scatter plot of 2000 points of the Pyramid Distribution between ± 3 standard deviations in each variable

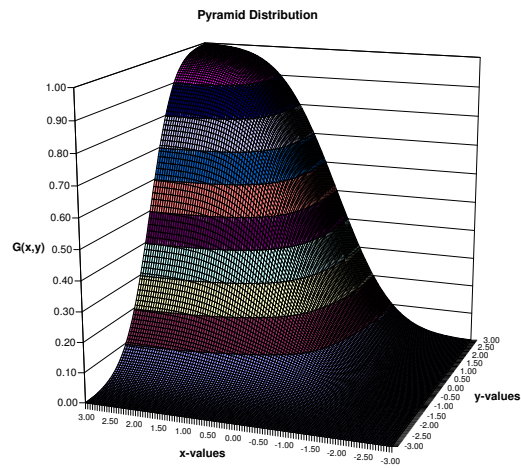


FIGURE 6. Pyramid Distribution Function

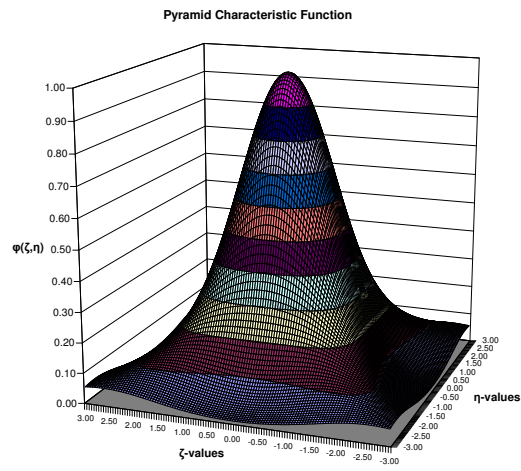


FIGURE 7. Pyramid Distribution Characteristic Function

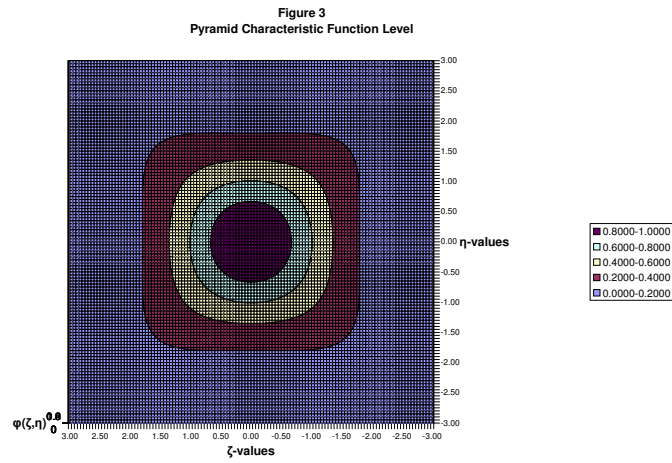


FIGURE 8. Pyramid Distribution Characteristic Function Level Curves

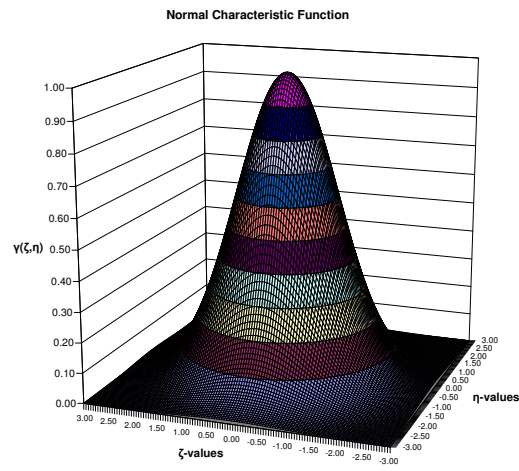


FIGURE 9. Normal Distribution Characteristic Function

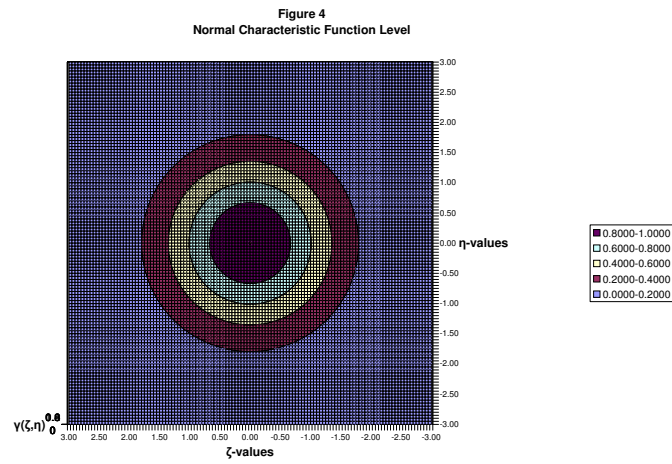


FIGURE 10. Normal Distribution Characteristic Function Level Curves

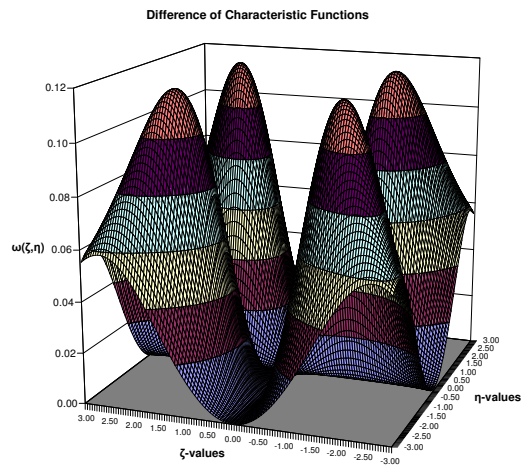


FIGURE 11. Difference of Pyramid and Normal Characteristic Functions

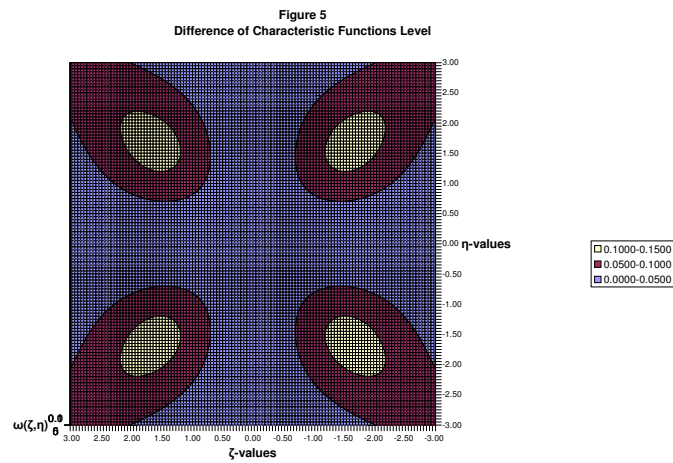


FIGURE 12. Difference of Pyramid and Normal Characteristic Function Level Curves

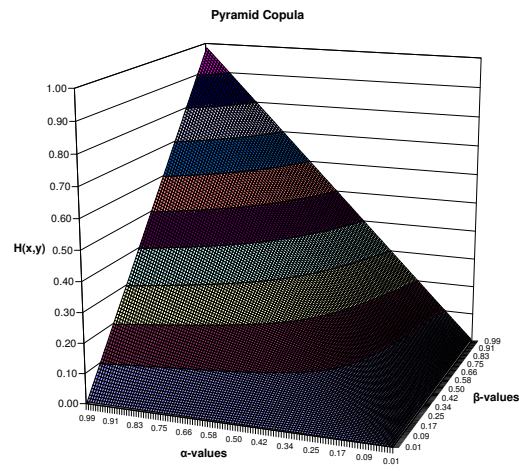


FIGURE 13. Pyramid Distribution Copula

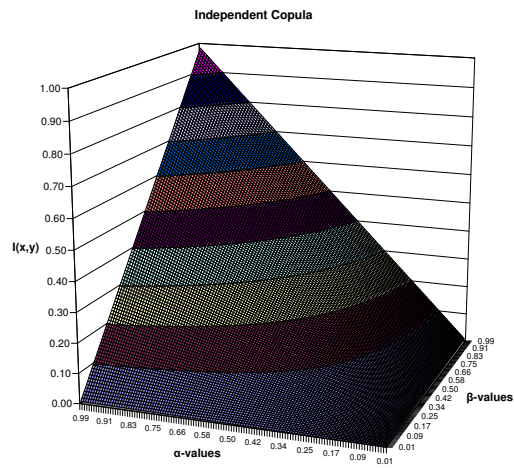


FIGURE 14. Independent Distribution (Product) Copula

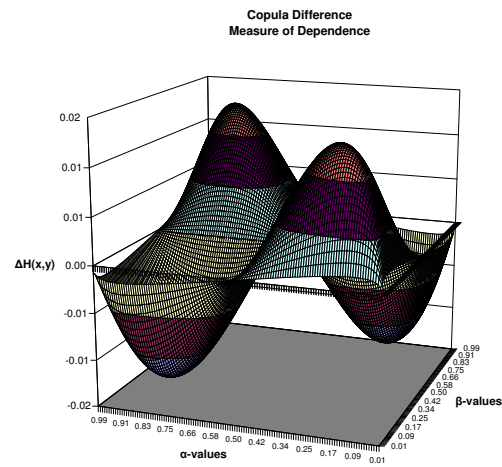


FIGURE 15. Difference of Pyramid and Independent Copulas

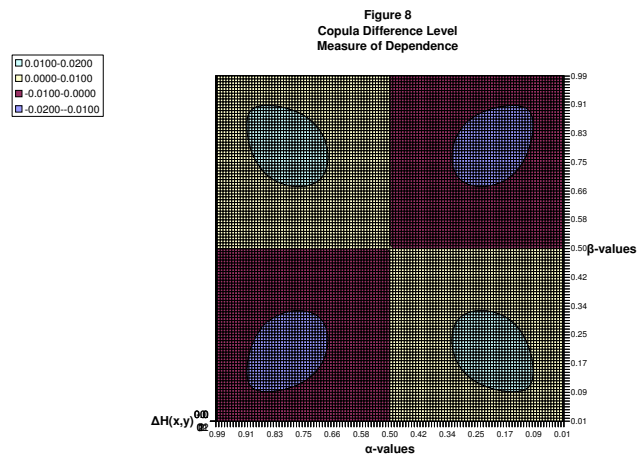


FIGURE 16. Difference of Pyramid and Independent Copula Level Curves

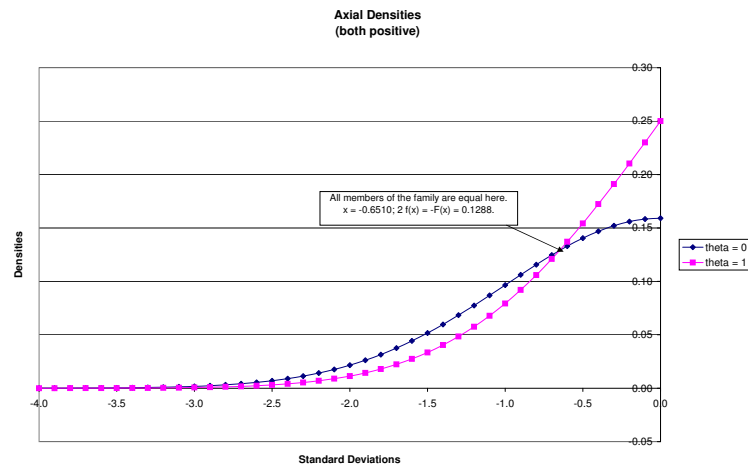


FIGURE 17. Axial Densities of the Pyramid Family, $\theta = +1$, $\theta = 0$, $\theta = -1$

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